

Ideal Projectors of Type Partial Derivative and Their Perturbations

Zhe Li^a, Shugong Zhang^b, Tian Dong^{*,b}

^aKey Laboratory of Mathematics Mechanization, AMSS, Beijing 100190, China

^bSchool of Mathematics, Key Lab. of Symbolic Computation and Knowledge Engineering (Ministry of Education), Jilin University, Changchun 130012, China

Abstract

In this paper, we verify Carl de Boor's conjecture on ideal projectors for real ideal projectors of type partial derivative by proving that there exists a positive $\eta \in \mathbb{R}$ such that a real ideal projector of type partial derivative P is the pointwise limit of a sequence of Lagrange projectors which are perturbed from P up to η in magnitude. Furthermore, we present an algorithm for computing the value of such η when the range of the Lagrange projectors is spanned by the Gröbner éscalier of their kernels w.r.t. lexicographic order.

Key words: Ideal projector, Ideal projector of type partial derivative, Carl de Boor's conjecture

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1. Introduction

Polynomial interpolation is to construct a polynomial p belonging to a finite-dimensional polynomial subspace from a set of data that agrees with a given function f at the data set. Univariate polynomial interpolation has a well developed theory, while the multivariate one is very problematic since a multivariate interpolation polynomial is determined not only by the cardinal but also by the geometry of the data set, cf. [1, 2].

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*Corresponding author

Email address: dongtian.jlu@gmail.com (Tian Dong)

As an elegant form of multivariate approximation, ideal interpolation provides a natural link between multivariate polynomial interpolation and algebraic geometry[3]. The study of ideal interpolation was initiated by Birkhoff [4] and continued by several authors [2, 5, 3, 6].

Actually, ideal interpolation is an *ideal projector* on polynomial ring whose kernel is an ideal. When the kernel of an ideal projector P is the vanishing ideal of certain finite nonempty set Ξ in \mathbb{R}^d , P is a *Lagrange projector* on $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_d]$, the polynomial ring in d variables over \mathbb{R} , which provides the Lagrange interpolation on Ξ . Obviously, P is finite-dimensional since its range is a $\#\Xi$ -dimensional subspace of $\mathbb{R}[\mathbf{x}]$. Lagrange projectors are standard examples of ideal projectors.

It is well-known that every univariate ideal projector is an *Hermite projector*, namely it is the pointwise limit of a sequence of Lagrange projectors. This inspired Carl de Boor[5] to conjecture that every finite-dimensional linear operator on $\mathbb{C}[\mathbf{x}]$ is an ideal projector if and only if it is Hermite.

However, Boris Shekhtman[7] disproved this conjecture when the dimension $d \geq 3$. In the same paper, Shekhtman also showed that the conjecture is true for bivariate complex projectors with the help of Fogarty Theorem (see [8]). Later, using linear algebra tools only, de Boor and Shekhtman[9] reproved the same result. Specifically, Shekhtman[10] completely analyzed the bivariate ideal projectors which are onto the space of polynomials of degree less than n over real or complex field, and verified the conjecture in this particular case.

Let P be an ideal projector that only interpolates a function and its partial derivatives. Obviously, many classical multivariate interpolation projectors are examples of P which has applications in many fields of mathematics and science, cf.[11]. Naturally, we wonder whether de Boor's conjecture is true for P or not.

In this paper, a positive answer is offered to this question by Theorem 2 of Section 3 which states that there exists a positive $\eta \in \mathbb{R}$ such that P is the pointwise limit of a sequence of Lagrange projectors which are perturbed from P up to η in magnitude, and the proof of the theorem is postponed to Section 5, the last section of the paper. A further natural question is how to determine the value of η . We propose an algorithm in Section 3 for computing the value of such η when the range of the Lagrange projectors is spanned by the Gröbner éscalier of their kernels w.r.t. lexicographic order. And then, Section 4 is dedicated to some examples to illustrate the algorithm. The next section, Section 2, is devoted as a preparation for this paper.

2. Preliminaries

In this section, we will introduce some notation and review some basic facts related to ideal projectors. For more details, we refer the reader to [5, 3, 12].

Throughout the paper, we use \mathbb{N}_0 to stand for the monoid of nonnegative integers and boldface type for tuples with their entries denoted by the same letter with subscripts, for example, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$.

Henceforward, we use \leq to denote the usual product order on \mathbb{N}_0^d , that is, for arbitrary $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^d$, $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if and only if $\alpha_i \leq \beta_i, i = 1, \dots, d$. A finite nonempty set $\mathfrak{d} \subset \mathbb{N}_0^d$ is called *lower* if for every $\boldsymbol{\alpha} \in \mathfrak{d}$, $\mathbf{0} \leq \boldsymbol{\beta} \leq \boldsymbol{\alpha}$ implies $\boldsymbol{\beta} \in \mathfrak{d}$.

A *monomial* $\mathbf{x}^\alpha \in \mathbb{R}[\mathbf{x}]$ is a power product of the form $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with $\boldsymbol{\alpha} \in \mathbb{N}_0^d$. Thus, a *polynomial* p in $\mathbb{R}[\mathbf{x}]$ can be expressed as a linear combination of monomials from $\text{Supp}(p)$, the support of p , as follows,

$$p = \sum_{\boldsymbol{\alpha}} \widehat{p}(\boldsymbol{\alpha}) \mathbf{x}^\alpha \quad (1)$$

where $\widehat{p}(\boldsymbol{\alpha}) \in \mathbb{R} \setminus \{0\}$. For $\mathbf{i} \in \mathbb{N}_0^d$ and $p \in \mathbb{R}[\mathbf{x}]$, if there exists a monomial $\mathbf{x}^{\boldsymbol{\alpha}'}$ in $\text{Supp}(p)$ such that $\boldsymbol{\alpha}' < \mathbf{i}$, then we denote this fact as $p <_m \mathbf{i}$.

Let P be a finite-dimensional ideal projector on $\mathbb{R}[\mathbf{x}]$. The range and the kernel of P are denoted by $\text{ran}P$ and $\ker P$ respectively. Furthermore, P has a dual projector P' on $\mathbb{R}'[\mathbf{x}]$, the algebraic dual of $\mathbb{R}[\mathbf{x}]$, whose range can be described as

$$\text{ran}P' = \{\lambda \in \mathbb{R}'[\mathbf{x}] : \ker P \subset \ker \lambda\},$$

which is the set of interpolation conditions matched by P . Assume that $\Lambda \subset \mathbb{R}'[\mathbf{x}]$ is an \mathbb{R} -basis for $\text{ran}P'$, then

$$\ker \Lambda := \{f \in \mathbb{R}[\mathbf{x}] : \lambda(f) = 0, \forall \lambda \in \Lambda\} = \ker P.$$

We denote by \mathbb{T}^d the monoid of all monomials in $\mathbb{R}[\mathbf{x}]$. For each fixed monomial order \prec on \mathbb{T}^d , a nonzero polynomial $f \in \mathbb{R}[\mathbf{x}]$ has a unique *leading monomial* $\text{LM}_\prec(f)$, which is the \prec -greatest monomial appearing in f with nonzero coefficient. According to [13], the monomial set

$$\mathcal{N}_\prec(\ker \Lambda) := \{\mathbf{x}^\alpha \in \mathbb{T}^d : \text{LM}_\prec(f) \nmid \mathbf{x}^\alpha, \forall f \in \ker \Lambda\}$$

is the *Gröbner éscalier* of $\ker \Lambda$ w.r.t. \prec . We denote by $\text{ran}_\prec P$ the range of P spanned by the Gröbner éscalier of $\ker \Lambda$ w.r.t. \prec .

When P is a Lagrange projector, we have $\ker \Lambda = \mathcal{I}(\Xi)$, the vanishing ideal of some finite nonempty set $\Xi \subset \mathbb{R}^d$. In 1995, Cerlienco and Mureddu[14] proposed an purely combinatorial algorithm named MB for computing the Gröbner éscalier of $\mathcal{I}(\Xi)$ w.r.t. some lexicographical order on \mathbb{T}^d which is denoted by \prec_{lex} here. Later, Felszeghy, Ráth, and Rónyai[15] provided a faster algorithm, lex game algorithm, by building a rooted tree $T(\Xi)$ of d levels from Ξ in the following way:

- The nodes on each path from the root to a leaf are labeled with the coordinates of a point.
- The root is regarded as the 0-th level with no label, its children are labeled with the d -th coordinates of the points, their children with the $(d-1)$ -coordinates, and so forth.
- If two points have same k ending coordinates, then their corresponding paths coincide until level k .

Given finite nonempty point sets $\Xi^{(1)}, \Xi^{(2)} \subset \mathbb{R}^d$ with $\#\Xi^{(1)} = \#\Xi^{(2)}$. If $T(\Xi^{(1)})$ and $T(\Xi^{(2)})$ have same structure, [15] showed that $\mathcal{N}_{\prec_{lex}}(\mathcal{I}(\Xi^{(1)})) = \mathcal{N}_{\prec_{lex}}(\mathcal{I}(\Xi^{(2)}))$.

3. Main results

Let

$$\delta_{\xi} : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R} : f \mapsto f(\xi)$$

denote the evaluation functional at the point $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, and let

$$D^{\alpha} : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}[\mathbf{x}] : f \mapsto \frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}} f := \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} f$$

be the differential operator with respect to $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ with $D^0 = I$, the identity operator on $\mathbb{R}[\mathbf{x}]$.

Definition 1. Let P be a finite-dimensional ideal projector on $\mathbb{R}[\mathbf{x}]$. If there exist distinct points $\xi^{(1)}, \dots, \xi^{(\mu)} \in \mathbb{R}^d$ and their associated lower sets $\mathfrak{d}^{(1)}, \dots, \mathfrak{d}^{(\mu)} \subset \mathbb{N}_0^d$ such that

$$\text{ran } P' = \text{Span}_{\mathbb{R}} \{ \delta_{\xi^{(k)}} \circ D^{\alpha} : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu \}, \quad (2)$$

namely P only interpolates a function and its partial derivatives, then we call P an *ideal projector of type partial derivative*.

As typical examples, Hermite projectors of type *total degree* and of type *coordinate degree* are both ideal projectors of type partial derivative, cf. [16].

Lemma 1. *Let $\xi^{(1)}, \dots, \xi^{(\mu)} \in \mathbb{R}^d$ be distinct points, and let $\mathfrak{d}^{(1)}, \dots, \mathfrak{d}^{(\mu)} \subset \mathbb{N}_0^d$ be their associated lower sets. Set*

$$\eta_0 := \min \left\{ \frac{\|\xi^{(k)} - \xi^{(l)}\|_2}{\|\alpha - \alpha'\|_2} : \alpha \in \mathfrak{d}^{(k)}, \alpha' \in \mathfrak{d}^{(l)}, \alpha \neq \alpha', 1 \leq k < l \leq \mu \right\}. \quad (3)$$

Then for arbitrary nonzero $h \in (-\eta_0, \eta_0) \subset \mathbb{R}$, the point set

$$\Xi_h := \{\xi^{(k)} + h\alpha : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu\} \quad (4)$$

exactly consists of $\# \sum_{i=1}^{\mu} \mathfrak{d}^{(i)}$ distinct points.

PROOF. Suppose that there exist $\alpha \in \mathfrak{d}^{(k)}$ and $\alpha' \in \mathfrak{d}^{(l)}$ with $1 \leq k < l \leq \mu$ such that $\xi^{(k)} + h\alpha = \xi^{(l)} + h\alpha'$ which implies that $\alpha \neq \alpha'$ by $\xi^{(k)} \neq \xi^{(l)}$. Consequently, we have

$$h = \frac{\|\xi^{(k)} - \xi^{(l)}\|_2}{\|\alpha - \alpha'\|_2},$$

which is in direct contradiction to the hypothesis that $0 < |h| < \eta_0$. \square

Lemma 1 holds out the possibility of intuitively perturbing an ideal projector of type partial derivative to a sequence of Lagrange projectors.

Definition 2. Let P be an ideal projector of type partial derivative on $\mathbb{R}[\mathbf{x}]$ with $\text{ran} P'$ described by (2). For an arbitrary fixed $h \in \mathbb{R}$ with $0 < |h| < \eta_0$ where η_0 is as in (3), define P_h to be the Lagrange projector on $\mathbb{R}[\mathbf{x}]$ with

$$\text{ran} P'_h = \text{Span}_{\mathbb{R}} \{\delta_{\xi^{(k)} + h\alpha} : \alpha \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu\}. \quad (5)$$

Then P_h is called an *h -perturbed Lagrange projector of P* .

Remark 1. It is easy to see from (2) and (5) that

$$\lambda := (\delta_{\xi^{(k)}} \circ D^\alpha : \alpha \in \mathfrak{d}^{(k)}, k = 1, \dots, \mu) \in (\mathbb{R}'[\mathbf{x}])^n$$

and

$$\lambda_h := (\delta_{\xi^{(k)} + h\alpha} : \alpha \in \mathfrak{d}^{(k)}, k = 1, \dots, \mu) \in (\mathbb{R}'[\mathbf{x}])^n$$

form \mathbb{R} -bases for $\text{ran}P'$ and $\text{ran}P'_h$ respectively, where $n = \sum_{k=1}^{\mu} \#\mathfrak{d}^{(k)}$.

Moreover, an ordering \prec_{λ} for the entries of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_h$ will be defined as follows: We say $\delta_{\boldsymbol{\xi}^{(k)}} \circ D^{\boldsymbol{\alpha}} \prec_{\lambda} \delta_{\boldsymbol{\xi}^{(k')}} \circ D^{\boldsymbol{\alpha}'}$ or $\delta_{\boldsymbol{\xi}^{(k)}+h\boldsymbol{\alpha}} \prec_{\lambda} \delta_{\boldsymbol{\xi}^{(k')}+h\boldsymbol{\alpha}'}$ if

$$k < k', \quad \text{or} \quad k = k' \text{ and } \boldsymbol{\alpha} \prec \boldsymbol{\alpha}',$$

where \prec is an arbitrary monomial order on \mathbb{N}_0^d .

We are now ready to give one of our main theorem, Theorem 2, which states that every ideal projector of type partial derivative on $\mathbb{R}[\mathbf{x}]$ is the pointwise limit of Lagrange projectors, namely Carl de Boor's conjecture is true for this type of ideal projectors.

Theorem 2. *Let P be an ideal projector of type partial derivative on $\mathbb{R}[\mathbf{x}]$ with $\text{ran}P'$ described by (2), and let $(P_h, 0 < |h| < \eta_0)$ be a sequence of perturbed Lagrange projector of P where η_0 is as in (3). Then the following statements hold:*

(i) *There exists a positive $\eta \in \mathbb{R}$ such that*

$$\text{ran}P_h = \text{ran}P, \quad \forall 0 < |h| < \eta \leq \eta_0.$$

(ii) *P is the pointwise limit of $P_h, 0 < |h| < \eta$, as h tends to zero.*

The proof of Theorem 2 will be provided in Section 5. Actually, with similar methodology there, we can easily prove the following theorem, which is a more general version of Theorem 2.

Theorem 3. *Let P be an ideal projector of type partial derivative from $C^\infty(\mathbb{R}^d)$ onto $\text{ran}P$, then there exists Lagrange projector P_h onto $\text{ran}P$ such that for all $f \in C^\infty(\mathbb{R}^d)$, Pf is the limit of $P_h f$ as h tends to zero.*

Now, after introducing Definition 3, we have an immediate corollary of Theorem 2.

Definition 3. [3] Let P be an ideal projector from $\mathbb{R}[\mathbf{x}]$ onto $\text{ran}P$ with $\dim \text{ran}P = n$. Assume that $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}[\mathbf{x}]^n$ is an \mathbb{R} -basis for $\text{ran}P$, and the border set $\partial\mathbf{q}$ of \mathbf{q} is defined by

$$\partial\mathbf{q} := \{1, x_k q_l, k = 1, \dots, d, l = 1, \dots, n\} \setminus \{q_1, \dots, q_n\}.$$

Then the set of polynomials

$$\{f - Pf : f \in \partial\mathbf{q}\}$$

forms a *border basis* for $\ker P$, which is called a \mathbf{q} -border basis for $\ker P$.

Corollary 4. *Let P be an ideal projector of type partial derivative on $\mathbb{R}[\mathbf{x}]$, and let \mathbf{q} be an \mathbb{R} -basis for $\text{ran}P$. Then there exists a Lagrange projector P_h onto $\text{ran}P$ such that the \mathbf{q} -border basis for $\ker P$ is the limit of \mathbf{q} -border basis for $\ker P_h$ as h tends to zero.*

Theorem 2 tells us that every ideal projector of type partial derivative is the pointwise limit of Lagrange projectors. Unfortunately, the converse statement is not true in general as the following example illustrates.

Example 1. Let $(P_h, 0 < |h| < 1)$ be a sequence of Lagrange projectors with

$$\begin{aligned}\text{ran}P_h &= \text{Span}_{\mathbb{R}}\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}, \\ \text{ran}P'_h &= \text{Span}_{\mathbb{R}}\{\delta_{(0,0)}, \delta_{(0,h)}, \delta_{(h,0)}, \delta_{(1,1)}, \delta_{(1,1+h)}, \delta_{(1+h,1)}\},\end{aligned}$$

and let P be an ideal projector with

$$\begin{aligned}\text{ran}P' &= \left\{ \delta_{(0,0)} \circ D^{(0,0)}, \delta_{(0,0)} \circ D^{(1,0)}, \delta_{(0,0)} \circ D^{(0,1)}, \right. \\ &\quad \left. \delta_{(1,1)} \circ D^{(0,0)}, \delta_{(1,1)} \circ D^{(1,0)}, \delta_{(1,1)} \circ D^{(0,1)} \right\}.\end{aligned}$$

However, $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ can not form an \mathbb{R} -basis for $\text{ran}P$. Hence, $(P_h, 0 < |h| < 1)$ can not converge pointwise to P , as h tends to zero.

Consider the bijection

$$\begin{aligned}u : \mathbb{R}^d \times \mathbb{N}_0^d &\longrightarrow (\mathbb{R} \times \mathbb{N}_0)^d \\ (\boldsymbol{\xi}, \boldsymbol{\alpha}) &\longrightarrow ((\xi_1, \alpha_1), \dots, (\xi_d, \alpha_d)).\end{aligned}$$

Let $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(\mu)} \in \mathbb{R}^d$ be distinct points and $\mathfrak{d}^{(1)}, \dots, \mathfrak{d}^{(\mu)} \subset \mathbb{N}_0^d$ be lower sets. Then

$$\Omega := \{u(\boldsymbol{\xi}^{(k)}, \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathfrak{d}^{(k)}, k = 1, \dots, \mu\} \subset (\mathbb{R} \times \mathbb{N}_0)^d \quad (6)$$

is called an *algebraic multiset*. As mentioned by [14], MB algorithm can be applied for the algebraic multiset Ω to obtain the Gröbner éscalier of the ideal

$$\{p \in \mathbb{R}[\mathbf{x}] : \delta_{\boldsymbol{\xi}^{(k)}} \circ D^{\boldsymbol{\alpha}}(p) = 0, \boldsymbol{\alpha} \in \mathfrak{d}^{(k)}, 1 \leq k \leq \mu\}$$

w.r.t. lexicographic order.

Recall Section 2. We have known how to build a d -level tree $T(\Xi)$ from a finite nonempty set $\Xi \in \mathbb{R}^d$. If the space \mathbb{R}^d is changed to $(\mathbb{R} \times \mathbb{N}_0)^d$, it is easy to see that we can also build a d -level tree $T(\Omega)$ from algebraic multiset Ω following the same rules, which makes lex game algorithm involved and leads to the following useful lemma.

Lemma 5. *Let P be an ideal projector of type partial derivative with $\text{ran} P'$ as in (2), and let P_h be a perturbed Lagrange projector of P . Let algebraic multiset $\Omega \subset (\mathbb{R} \times \mathbb{N}_0)^d$ be as in (6) and $\Xi_h \subset \mathbb{R}^d$ be as in (4). If the rooted trees $T(\Omega)$ and $T(\Xi_h)$ have the same structure, then*

$$\text{ran}_{\prec_{lex}} P = \text{ran}_{\prec_{lex}} P_h.$$

Next, we can proceed with another main theorem of this paper.

Theorem 6. *Let P be an ideal projector of type partial derivative with $\text{ran} P'$ as in (2), and let $(P_h, 0 < |h| < \eta)$ be a sequence of h -perturbed Lagrange projectors of P , where η is obtained through Algorithm 1 in the following. If the range of P_h is $\text{ran}_{\prec_{lex}} P_h$, then the sequence $(P_h, 0 < |h| < \eta)$ converges pointwise to the ideal projector P , as h tends to zero.*

Algorithm 1. (The range for $|h|$)

Input: Distinct points $\xi^{(1)}, \dots, \xi^{(\mu)} \in \mathbb{R}^d$ and lower sets $\mathfrak{d}^{(1)}, \dots, \mathfrak{d}^{(\mu)} \subset \mathbb{N}_0^d$.

Output: A nonnegative number $\eta \in \mathbb{R}$ or ∞ .

Step 1 Construct algebraic multiset Ω from $\xi^{(1)}, \dots, \xi^{(\mu)}$ and $\mathfrak{d}^{(1)}, \dots, \mathfrak{d}^{(\mu)}$ following (6), and then build rooted tree $T(\Omega)$ from Ω in the way introduced in Section 2.

Step 2 Suppose that the first level nodes of $T(\Omega)$ are labeled with the points of set $\mathcal{L}_1 \subset \mathbb{R} \times \mathbb{N}_0$.

Step 2.1 If $\#\mathcal{L}_1 = 1$, then $\eta \leftarrow \infty$.

Step 2.2 If every point in \mathcal{L}_1 has the same first coordinate or the same second coordinate, then $\eta \leftarrow \infty$.

Step 2.3: Otherwise, set

$$\eta \leftarrow \min \left\{ \frac{|\xi_d^{(i)} - \xi_d^{(j)}|}{|\alpha_d^{(i)} - \alpha_d^{(j)}|} : \xi_d^{(i)} \neq \xi_d^{(j)}, \alpha_d^{(i)} \neq \alpha_d^{(j)}, \right. \\ \left. (\xi_d^{(i)}, \alpha_d^{(i)}) \text{ and } (\xi_d^{(j)}, \alpha_d^{(j)}) \in \mathcal{L}_1 \right\}.$$

Step 3 Set $k \rightarrow 2$.

Step 4 Suppose that the k -th level nodes are labeled respectively with the points of sets $\mathcal{L}_k^{(1)}, \dots, \mathcal{L}_k^{(\nu)} \subset \mathbb{R} \times \mathbb{N}_0$, where for each $1 \leq l \leq \nu$, the nodes labeled with the points in $\mathcal{L}_k^{(l)}$ share the same parent. For $l = 1, \dots, \nu$ and $\#\mathcal{L}_k^{(l)} \geq 2$, do the following steps.

Step 4.1 Set

$$\eta' \leftarrow \min \left\{ \frac{|\xi_{d-k+1}^{(i)} - \xi_{d-k+1}^{(j)}|}{|\alpha_{d-k+1}^{(i)} - \alpha_{d-k+1}^{(j)}|} : \xi_{d-k+1}^{(i)} \neq \xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(i)} \neq \alpha_{d-k+1}^{(j)}, \right. \\ \left. (\xi_{d-k+1}^{(i)}, \alpha_{d-k+1}^{(i)}) \text{ and } (\xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(j)}) \in \mathcal{L}_k^{(l)} \right\}.$$

Step 4.2 If $\eta' < \eta$, then $\eta \leftarrow \eta'$.

Step 5 If $k = d$, then return η and stop. Otherwise set $k \leftarrow k + 1$, continue with Step 4.

PROOF. To prove this theorem, by Lemma 5 and Theorem 2, it suffices to show that the rooted trees $T(\Xi_h), 0 < |h| < \eta$, and $T(\Omega)$ have the same structure, where Ξ_h is as in (4) and Ω is as in (6). Now, with the notation in Algorithm 1, we will use induction on the number of levels k of the rooted tree to prove this.

When $k = 1$, assume that there exist some $(\xi_d^{(i)}, \alpha_d^{(i)})$ and $(\xi_d^{(j)}, \alpha_d^{(j)}) \in \mathcal{L}_1$ such that $\xi_d^{(i)} + h\alpha_d^{(i)} = \xi_d^{(j)} + h\alpha_d^{(j)}$. The same argument in Lemma 1 shows that $h = |\xi_d^{(i)} - \xi_d^{(j)}|/|\alpha_d^{(i)} - \alpha_d^{(j)}|$ where $\alpha_d^{(i)} \neq \alpha_d^{(j)}$ and $\xi_d^{(i)} \neq \xi_d^{(j)}$, which contradicts

$$|h| < \min \left\{ \frac{|\xi_d^{(i)} - \xi_d^{(j)}|}{|\alpha_d^{(i)} - \alpha_d^{(j)}|} : \xi_d^{(i)} \neq \xi_d^{(j)}, \alpha_d^{(i)} \neq \alpha_d^{(j)}, \right. \\ \left. (\xi_d^{(i)}, \alpha_d^{(i)}) \text{ and } (\xi_d^{(j)}, \alpha_d^{(j)}) \in \mathcal{L}_1 \right\}.$$

Hence, the first levels of $T(\Xi_h), 0 < |h| < \eta$, and $T(\Omega)$ have the same structure.

Suppose that the first $k - 1$ levels of $T(\Xi_h), 0 < |h| < \eta$, and $T(\Omega)$ have the same structure. Assume that there exists some $1 \leq l \leq \nu$ and $(\xi_{d-k+1}^{(i)}, \alpha_{d-k+1}^{(i)}), (\xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(j)}) \in \mathcal{L}_k^{(l)}$ such that $\xi_{d-k+1}^{(i)} + h\alpha_{d-k+1}^{(i)} = \xi_{d-k+1}^{(j)} +$

$h\alpha_{d-k+1}^{(j)}$. Since $(\xi_{d-k+1}^{(i)}, \alpha_{d-k+1}^{(i)})$, $(\xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(j)})$ have common parent, it is easy to see that $h = |\xi_{d-k+1}^{(i)} - \xi_{d-k+1}^{(j)}|/|\alpha_{d-k+1}^{(i)} - \alpha_{d-k+1}^{(j)}|$ where $\alpha_{d-k+1}^{(i)} \neq \alpha_{d-k+1}^{(j)}$ and $\xi_{d-k+1}^{(i)} \neq \xi_{d-k+1}^{(j)}$, which contradicts the fact

$$|h| < \min \left\{ \frac{|\xi_{d-k+1}^{(i)} - \xi_{d-k+1}^{(j)}|}{|\alpha_{d-k+1}^{(i)} - \alpha_{d-k+1}^{(j)}|} : \xi_{d-k+1}^{(i)} \neq \xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(i)} \neq \alpha_{d-k+1}^{(j)} \right. \\ \left. (\xi_{d-k+1}^{(i)}, \alpha_{d-k+1}^{(i)}) \text{ and } (\xi_{d-k+1}^{(j)}, \alpha_{d-k+1}^{(j)}) \in \mathcal{L}_k^{(l)} \right\}.$$

Therefore, the first k levels of $T(\Xi_h)$, $0 < |h| < \eta$, and $T(\Omega)$ have the same structure. \square

4. Example

In this section, we will present several examples to illustrate Theorem 6.

Example 2. Assume that P_h is a Lagrange projector with

$$\text{ran} P'_h = \text{Span}_{\mathbb{R}} \{ \delta_{(0,0)}, \delta_{(h,0)}, \delta_{(0,h)}, \delta_{(1,1)}, \delta_{(1+h,1)}, \delta_{(1,1+h)} \}.$$

Construct the rooted tree of the algebraic multiset

$$\Omega = \{ ((0,0), (0,0)), ((0,1), (0,0)), ((0,0), (0,1)), \\ ((1,0), (1,0)), ((1,1), (1,0)), ((1,0), (1,1)) \}.$$

$T(\Omega)$ is illustrated in Figure 1. By Algorithm 1, we obtain $\eta = 1$. From Theorem 6, we can conclude that $(P_h, 0 < |h| < 1)$ onto $\text{Span}_{\mathbb{R}} \{1, x_2, x_1, x_2^2, x_1x_2, x_2^3\}$ pointwise converges to an Hermite projector P with

$$\text{ran} P' = \{ \delta_{(0,0)} \circ D^{(0,0)}, \delta_{(0,0)} \circ D^{(1,0)}, \delta_{(0,0)} \circ D^{(0,1)}, \delta_{(1,1)} \circ D^{(0,0)}, \delta_{(1,1)} \circ D^{(1,0)}, \\ \delta_{(1,1)} \circ D^{(0,1)} \},$$

as h tends to zero.

Example 3. Assume that P_h is a Lagrange projector with

$$\text{ran} P'_h = \text{Span}_{\mathbb{R}} \{ \delta_{(0,0,0)}, \delta_{(h,0,0)}, \delta_{(0,h,0)}, \delta_{(0,0,h)}, \delta_{(1,1,1)}, \\ \delta_{(1+h,1,1)}, \delta_{(1,1+h,1)}, \delta_{(1,1,1+h)} \}.$$

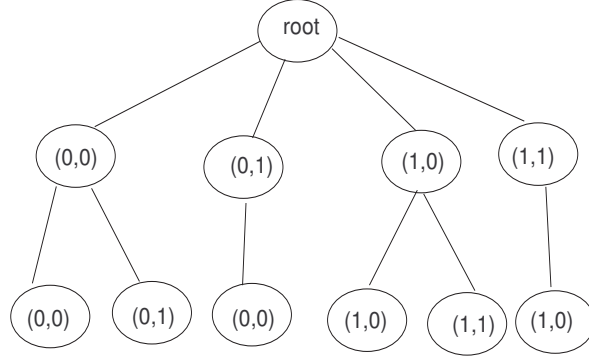


Figure 1: $T(\Omega)$ of Example 2

Construct the rooted tree of the algebraic multiset

$$\Omega = \{((0,0), (0,0), (0,0)), ((0,1), (0,0), (0,0)), ((0,0), (0,1), (0,0)), ((0,0), (0,0), (0,1)), ((1,0), (1,0), (1,0)), ((1,1), (1,0), (1,0)), ((1,0), (1,1), (1,0)), ((1,0), (1,0), (1,1))\}.$$

$T(\Omega)$ is illustrated in Figure 2. By Algorithm 1, we compute $\eta = 1$. From

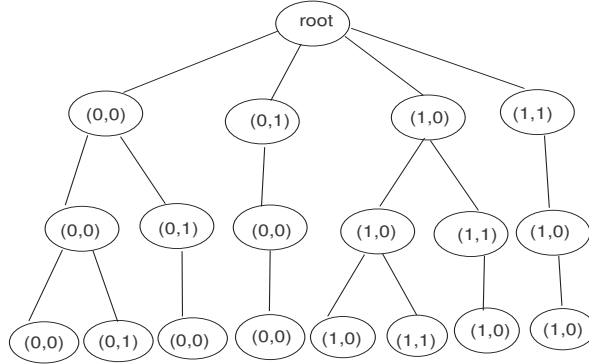


Figure 2: $T(\Omega)$ of Example 3

Theorem 6, we can conclude that $(P_h, 0 < |h| < 1)$ onto $\{1, x_3, x_2, x_1, x_3^2, x_2x_3,$

$x_1x_3, x_3^3\}$ pointwise converges to an Hermite projector P with

$$\begin{aligned} \text{ran} P' = \{ & \delta_{(0,0,0)} \circ D^{(0,0,0)}, \delta_{(0,0,0)} \circ D^{(1,0,0)}, \delta_{(0,0,0)} \circ D^{(0,1,0)}, \delta_{(0,0,0)} \circ D^{(0,0,1)}, \\ & \delta_{(1,1,1)} \circ D^{(0,0,0)}, \delta_{(1,1,1)} \circ D^{(1,0,0)}, \delta_{(1,1,1)} \circ D^{(0,1,0)}, \delta_{(1,1,1)} \circ D^{(0,0,1)} \}, \end{aligned}$$

as h tends to zero.

Finally, we select test functions

$$\begin{aligned} f_1(x_1, x_2) &= 1 + (1 - x_1)^4 + (1 - x_2)^4, \\ f_2(x_1, x_2, x_3) &= 1 + (1 - x_1)^2 + (1 - x_2)^2 + (1 - x_3)^2, \end{aligned}$$

to illustrate the pointwise convergence of ideal projectors of type partial derivative in the above examples.

For Example 2, when $h = 1/10, 1/100, 1/1000, \dots$, we have

$$\begin{aligned} P_{\frac{1}{10}} f_1 &= 3 - \frac{385039}{99000} x_2 - \frac{3439}{1000} x_1 + \frac{719}{150} x_2^2 + \frac{86}{25} x_1 x_2 - \frac{1438}{495} x_2^3, \\ P_{\frac{1}{100}} f_1 &= 3 - \frac{39984109399}{9999000000} x_2 - \frac{3940399}{1000000} x_1 + \frac{970199}{165000} x_2^2 + \frac{9851}{2500} x_1 x_2 \\ &\quad - \frac{970199}{249975} x_2^3, \\ P_{\frac{1}{1000}} f_1 &= 3 - \frac{571426287284857}{142857000000000} x_2 - \frac{3994003999}{1000000000} x_1 + \frac{997001999}{166500000} x_2^2 \\ &\quad + \frac{998501}{250000} x_1 x_2 - \frac{142428857}{35714250} x_2^3, \\ &\dots \\ P f_1 &= 3 - 4x_2 - 4x_1 + 6x_2^2 + 4x_1 x_2 - 4x_2^3. \end{aligned}$$

For Example 3,

$$\begin{aligned}
P_{\frac{1}{10}} f_2 &= 4 - \frac{829}{495} x_3 - \frac{19}{10} x_2 - \frac{19}{10} x_1 - \frac{7}{3} x_3^2 + 2x_2 x_3 + 2x_1 x_3 + \frac{80}{99} x_3^3, \\
P_{\frac{1}{100}} f_3 &= 4 - \frac{989299}{499950} x_3 - \frac{199}{100} x_2 - \frac{199}{100} x_1 - \frac{37}{33} x_3^2 + 2x_2 x_3 + 2x_1 x_3 \\
&\quad + \frac{800}{9999} x_3^3, \\
P_{\frac{1}{1000}} f_2 &= 4 - \frac{998992999}{499999500} x_3 - \frac{1999}{1000} x_2 - \frac{1999}{1000} x_1 - \frac{337}{333} x_3^2 + 2x_2 x_3 + 2x_1 x_3 \\
&\quad + \frac{8000}{999999} x_3^3, \\
&\quad \dots \\
P f_2 &= 4 - 2x_3 - 2x_2 - 2x_1 - x_3^2 + 2x_2 x_3 + 2x_1 x_3.
\end{aligned}$$

5. Proof of Theorem 2

First of all, we need to relate forward differences of multivariate polynomials to their partial derivatives. The following formula is quite useful for this purpose.

Lemma 7. *Let $i, m \in \mathbb{N}_0$ satisfying $i \geq m > 0$. Then*

$$\sum_{j=0}^{i-1} (-1)^j \binom{i}{j} (i-j)^m = \begin{cases} i!, & m = i; \\ 0, & m < i. \end{cases} \quad (7)$$

PROOF. The proof can be completed by induction on m . \square

Lemma 8. *Let $\xi, h \in \mathbb{R}, h \neq 0$, and $i, \alpha \in \mathbb{N}_0$. Then for every monomial x^α in $\mathbb{R}[x]$,*

$$\sum_{j=0}^i (-1)^j \binom{i}{j} \delta_{\xi+h(i-j)} x^\alpha = \begin{cases} h^i \delta_\xi \circ D^i x^\alpha, & \alpha \leq i; \\ h^i \delta_\xi \circ D^i x^\alpha + O(h^{i+1}), & \alpha > i, \end{cases} \quad (8)$$

where the remainder $O(h^{i+1})$ is a polynomial in h .

PROOF. From the theory of finite difference (see for example [17]) we know that

$$\Delta^i \delta_\xi f(x) = \sum_{j=0}^i (-1)^j \binom{i}{j} \delta_{\xi+h(i-j)} f(x) = h^i \delta_\xi \circ D^i f(x) + O(h^{i+1}),$$

where Δ is the forward difference operator and $f(x) \in C^i(\mathbb{R})$. When $f(x)$ is substituted by x^α in this equation, (8) follows immediately. Moreover, by Lemma 7, we can easily check that the remainder $O(h^{i+1})$ in (8) is a polynomial in h . This completes the proof. \square

The conclusion of Lemma 8 will be carried over to multivariate cases as follows.

Lemma 9. *Suppose that $h \in \mathbb{R} \setminus \{0\}$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, and $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{N}_0^d$. Then for arbitrary monomial \mathbf{x}^α in $\mathbb{R}[\mathbf{x}]$, we have*

$$\sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^{\mathbf{j}} \binom{\mathbf{i}}{\mathbf{j}} \delta_{\boldsymbol{\xi} + h(\mathbf{i} - \mathbf{j})} \mathbf{x}^\alpha = \begin{cases} h^{\|\mathbf{i}\|_1} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^\alpha + O(h^{\|\mathbf{i}\|_1 + 1}), & \mathbf{i} < \boldsymbol{\alpha}; \\ h^{\|\mathbf{i}\|_1} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^\alpha, & \text{otherwise,} \end{cases} \quad (9)$$

where $(-1)^{\mathbf{j}} = (-1)^{j_1} \dots (-1)^{j_d}$ and $\binom{\mathbf{i}}{\mathbf{j}} = \binom{i_1}{j_1} \dots \binom{i_d}{j_d}$ provided that $\mathbf{j} = (j_1, \dots, j_d)$.

PROOF. First, it follows from Lemma 8 that for every $1 \leq k \leq d$

$$\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k + h(i_k - j_k)} x_k^{\alpha_k} = \begin{cases} h^{i_k} \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k}, & \alpha_k \leq i_k; \\ h^{i_k} \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k} + O(h^{i_k+1}), & \alpha_k > i_k. \end{cases} \quad (10)$$

Further, we observe that

$$\prod_{k=1}^d \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k} = \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^\alpha \quad (11)$$

and

$$\sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^{\mathbf{j}} \binom{\mathbf{i}}{\mathbf{j}} \delta_{\boldsymbol{\xi} + h(\mathbf{i} - \mathbf{j})} \mathbf{x}^\alpha = \prod_{k=1}^d \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k + h(i_k - j_k)} x_k^{\alpha_k} \right). \quad (12)$$

Finally, we distinguish three cases to prove that the right-hand sides of (12) and (9) are equal to each other, which will complete the proof.

Case 1: $\boldsymbol{\alpha} \leq \mathbf{i}$.

Using (10) and (11), it is straightforward to verify that

$$\prod_{k=1}^d \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k + h(i_k - j_k)} x_k^{\alpha_k} \right) = \prod_{k=1}^d h^{i_k} \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k} = h^{\|\mathbf{i}\|_1} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^\alpha.$$

Case 2: $\mathbf{i} \not\leq \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \not\leq \mathbf{i}$.

In this case, there must exist some $1 \leq k, l \leq d$ such that $\alpha_k < i_k$ and $i_l < \alpha_l$. Thus, it is easily checked that

$$\prod_{k=1}^d \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k+h(i_k-j_k)} x_k^{\alpha_k} \right) = h^{\|\mathbf{i}\|_1} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^{\boldsymbol{\alpha}} = 0.$$

Case 3: $\mathbf{i} < \boldsymbol{\alpha}$.

Let $l = \max\{k : i_k < \alpha_k, 1 \leq k \leq d\}$. Then, applying (10) and (11), we deduce that

$$\begin{aligned} & \prod_{k=1}^d \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k+h(i_k-j_k)} x_k^{\alpha_k} \right) \\ &= \prod_{k=1}^l \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k+h(i_k-j_k)} x_k^{\alpha_k} \right) \prod_{k=l+1}^d \left(\sum_{j_k=0}^{i_k} (-1)^{j_k} \binom{i_k}{j_k} \delta_{\xi_k+h(i_k-j_k)} x_k^{\alpha_k} \right) \\ &= \prod_{k=1}^l (h^{i_k} \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k} + O(h^{i_k+1})) \prod_{k=l+1}^d h^{i_k} \delta_{\xi_k} \circ D^{i_k} x_k^{\alpha_k} \\ &= h^{\|\mathbf{i}\|_1} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} \mathbf{x}^{\boldsymbol{\alpha}} + O(h^{\|\mathbf{i}\|_1+1}), \end{aligned}$$

where the empty product is understood to be 1. \square

Equation (9) makes a connection between the forward difference calculus and the differential calculus for multivariate monomials. From Lemma 8, it follows that the remainder $O(h^{\|\mathbf{i}\|_1+1})$ in (9) is a polynomial in h . Equipped with these facts, we can establish the relationship between forward differences and partial derivatives of multivariate polynomials, which plays an important role in the proof of Theorem 2.

Corollary 10. *Let $\mathbf{i}, h, \boldsymbol{\xi}$ be as in Lemma 9 and $p \in \mathbb{R}[\mathbf{x}] \setminus \{0\}$. Then*

$$\frac{1}{h^{\|\mathbf{i}\|_1}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\boldsymbol{\xi}+h(\mathbf{i}-\mathbf{j})} p = \begin{cases} \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} p + O(h), & p <_m \mathbf{i}; \\ \delta_{\boldsymbol{\xi}} \circ D^{\mathbf{i}} p, & \text{otherwise.} \end{cases} \quad (13)$$

PROOF. Assume that nonzero polynomial p has form (1). Since

$$\sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\boldsymbol{\xi}+h(\mathbf{i}-\mathbf{j})} p = \sum_{\boldsymbol{\alpha}} \widehat{p}(\boldsymbol{\alpha}) \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\boldsymbol{\xi}+h(\mathbf{i}-\mathbf{j})} \mathbf{x}^{\boldsymbol{\alpha}}$$

and

$$\delta_{\xi} \circ D^i p = \sum_{\alpha} \widehat{p}(\alpha) \delta_{\xi} \circ D^i \mathbf{x}^{\alpha},$$

we get

$$\sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\xi+h(\mathbf{i}-\mathbf{j})} p = \begin{cases} h^{\|\mathbf{i}\|_1} \delta_{\xi} \circ D^i p + O(h^{\|\mathbf{i}\|_1+1}), & p <_m \mathbf{i}; \\ h^{\|\mathbf{i}\|_1} \delta_{\xi} \circ D^i p, & \text{otherwise,} \end{cases}$$

which leads to the corollary immediately. \square

Now, we are ready to prove Theorem 2.

PROOF OF THEOREM 2. We adopt the notation of Definition 1 and Remark 1. Let $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be an \mathbb{R} -basis for $\text{ran} P$. Without loss of generality, we assume that the entries of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_h$ are ordered ascendingly w.r.t. \prec_{λ} and then denoted as o_1, \dots, o_n and o'_1, \dots, o'_n respectively. For convenience, we set $n \times n$ matrices

$$\boldsymbol{\lambda}^T \mathbf{q} = (o_i q_j)_{1 \leq i, j \leq n}, \quad \boldsymbol{\lambda}_h^T \mathbf{q} = (o'_i q_j)_{1 \leq i, j \leq n},$$

and, therefore, for every $q \in \mathbb{R}[\mathbf{x}]$, n by 1 vectors

$$\boldsymbol{\lambda}^T q = (o_i q)_{1 \leq i \leq n}, \quad \boldsymbol{\lambda}_h^T q = (o'_i q)_{1 \leq i \leq n}.$$

By Corollary 10, equation (13) can be rewritten as

$$\delta_{\xi} \circ D^i p = \begin{cases} \frac{1}{h^{\|\mathbf{i}\|_1}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\xi+h(\mathbf{i}-\mathbf{j})} p + O(h), & p <_m \mathbf{i}; \\ \frac{1}{h^{\|\mathbf{i}\|_1}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{i}} (-1)^j \binom{\mathbf{i}}{\mathbf{j}} \delta_{\xi+h(\mathbf{i}-\mathbf{j})} p, & \text{otherwise,} \end{cases}$$

which implies that for fixed $1 \leq k \leq \mu$ and $\mathbf{i} \in \mathfrak{d}^{(k)}$, $\delta_{\xi^{(k)}} \circ D^i p$ can be linearly expressed by $\{\delta_{\xi^{(k)}+h\mathbf{l}} p : \mathbf{l} \in \mathfrak{d}^{(k)}\} \cup \{O(h)\}$ since $\mathfrak{d}^{(k)}$ is lower, and moreover, the linear combination coefficient of each $\delta_{\xi^{(k)}+h\mathbf{l}} p$ is independent of $p \in \mathbb{R}[\mathbf{x}]$. Thus, it turns out that there exists a nonsingular matrix T_p of order n such that

$$\left[\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \middle| \widehat{\boldsymbol{\lambda}^T \mathbf{q}} \right] := T_p \left[\boldsymbol{\lambda}_h^T \mathbf{q} \middle| \boldsymbol{\lambda}^T \mathbf{q} \right] = \left[\boldsymbol{\lambda}^T \mathbf{q} \middle| \boldsymbol{\lambda}^T \mathbf{q} \right] + [E_h \middle| \boldsymbol{\epsilon}_h], \quad (14)$$

where each entry of $[E_h \middle| \boldsymbol{\epsilon}_h]$ is either 0 or $O(h)$. As a consequence, the linear systems

$$\left(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \quad \text{and} \quad \left(\boldsymbol{\lambda}_h^T \mathbf{q} \right) \mathbf{x} = \boldsymbol{\lambda}_h^T \mathbf{q}$$

are equivalent, namely they have the same set of solutions.

(i) From (14), it follows that each entry of matrix $\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}}$ converges to its corresponding entry of matrix $\boldsymbol{\lambda}^T \mathbf{q}$ as h tends to zero, which implies that

$$\lim_{h \rightarrow 0} \det \left(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \right) = \det \left(\boldsymbol{\lambda}^T \mathbf{q} \right).$$

Since $\det(\boldsymbol{\lambda}^T \mathbf{q}) \neq 0$, there exists $\eta > 0$ such that

$$\det \left(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \right) \neq 0, \quad 0 < |h| < \eta.$$

Notice that (14) directly leads to $\text{rank} \left(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \right) = \text{rank} \left(\boldsymbol{\lambda}_h^T \mathbf{q} \right)$,

$$\text{ran} P_h = \text{Span}_{\mathbb{R}} \mathbf{q}, \quad 0 < |h| < \eta,$$

follows, i.e., \mathbf{q} forms an \mathbb{R} -basis for $\text{ran} P_h$. Since \mathbf{q} is also a basis for $\text{ran} P$, we have

$$\text{ran} P = \text{ran} P_h, \quad 0 < |h| < \eta.$$

(ii) Suppose that $\tilde{\mathbf{x}}_h$ and $\tilde{\mathbf{x}}$ be the unique solutions of nonsingular linear systems

$$(\boldsymbol{\lambda}_h^T \mathbf{q}) \mathbf{x} = \boldsymbol{\lambda}_h^T \mathbf{q} \tag{15}$$

and

$$(\boldsymbol{\lambda}^T \mathbf{q}) \mathbf{x} = \boldsymbol{\lambda}^T \mathbf{q} \tag{16}$$

respectively, where $0 < |h| < \eta$. It is easy to see that

$$P_h \mathbf{q} = \mathbf{q} \tilde{\mathbf{x}}_h \quad \text{and} \quad P \mathbf{q} = \mathbf{q} \tilde{\mathbf{x}}.$$

Remark that, as $h \rightarrow 0$, P is the pointwise limit of P_h if and only if $P \mathbf{q}$ is the coefficientwise limit of $P_h \mathbf{q}$ for all $\mathbf{q} \in \mathbb{R}[\mathbf{x}]$. Therefore, it is sufficient to show that for every $\mathbf{q} \in \mathbb{R}[\mathbf{x}]$, the solution vector of system (15) converges to the one of system (16) when h tends to zero, namely

$$\lim_{h \rightarrow 0} \tilde{\mathbf{x}}_h = \tilde{\mathbf{x}}.$$

By (14), the linear system

$$\left(\widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \right) \mathbf{x} = \widehat{\boldsymbol{\lambda}_h^T \mathbf{q}} \tag{17}$$

can be rewritten as

$$(\boldsymbol{\lambda}^T \mathbf{q} + E_h) \mathbf{x} = (\boldsymbol{\lambda}^T \mathbf{q} + \boldsymbol{\epsilon}_h).$$

Since system (17) is equivalent to system (15), $\tilde{\mathbf{x}}_h$ is also the unique solution of it. Consequently, applying the perturbation analysis of the sensitivity of linear systems (see for example [18], p.80ff), we have

$$\|\tilde{\mathbf{x}}_h - \tilde{\mathbf{x}}\| \leq \left\| (\boldsymbol{\lambda}^T \mathbf{q})^{-1} \right\| \|\boldsymbol{\epsilon}_h - E_h \tilde{\mathbf{x}}\| + O(h^2).$$

Since each entry of vector $\boldsymbol{\epsilon}_h - E_h \tilde{\mathbf{x}}$ is either 0 or $O(h)$, it follows that $\lim_{h \rightarrow 0} \|\tilde{\mathbf{x}}_h - \tilde{\mathbf{x}}\| = 0$, or, equivalently, $\lim_{h \rightarrow 0} \tilde{\mathbf{x}}_h = \tilde{\mathbf{x}}$, which completes the proof of the theorem. \square

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